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CERTAIN HYPERBOLIC CURVES OF THE nth ORDER.*

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1. In a plane are given n straight lines with the Cartesian equations

$$
\begin{aligned}
y&=\alpha_1 x+b_1,\\
y&=\alpha_2 x+b_2,\\
&\dots\dots\dots\\
y&=\alpha_n x+b_n.
\end{aligned}
$$

A straight line with the equation

$$y=\lambda x$$

passes through the origin of coördinates and intersects the n lines in n points. In this manner n segments, measured from the origin, are obtained for each position of this variable ray through the origin. The algebraic sum of these n segments taken on this same line will determine a point, P, which will describe a curve of the nth order as λ varies from $+\infty$ to $-\infty$.

The point of intersection of any one of the given lines with the variable ray is

$$x=\frac{b_n}{\lambda-\alpha_n}, \qquad y=\frac{b_n\lambda}{\lambda-\alpha_n};$$

*Summary of a thesis submitted to the Graduate Faculty of the University of Colorado for the degree of M. S., June, 1901.

therefore the coördinates (ξ, η), of the point P, are

$$\xi = \sum_1^n \frac{b_n}{\lambda - \alpha_n},$$

$$\eta = \sum_1^n \frac{b_n \lambda}{\lambda - \alpha_n}; \text{ or}$$

$$\xi = \frac{b_1}{\lambda - \alpha_1} + \frac{b_2}{\lambda - \alpha_2} + \ldots\ldots + \frac{b_n}{\lambda - \alpha_n},$$

$$\eta = \frac{b_1 \lambda}{\lambda - \alpha_1} + \frac{b_2 \lambda}{\lambda - \alpha_2} + \ldots\ldots + \frac{b_n \lambda}{\lambda - \alpha_n} = \lambda \xi.$$

From the last equation

$$\lambda = \frac{n}{\xi}.$$

Substituting this value of λ in the expression for ξ and simplifying the result is

$$\begin{aligned} \prod_1^n (\eta - \xi \alpha_n) = {} & b_1 (\eta - \xi \alpha_2)(\eta - \xi \alpha_3) \ldots\ldots (\eta - \xi \alpha_n) + \\ & b_2 (\eta - \xi \alpha_1)(\eta - \xi \alpha_3) \ldots\ldots (\eta - \xi \alpha_n) + \\ & \vdots \qquad \vdots \qquad \vdots \qquad \qquad \vdots \\ & b_n (\eta - \xi \alpha_1)(\eta - \xi \alpha_2) \ldots\ldots (\eta - \xi \alpha_{n-1}). \end{aligned}$$

This equation is of the nth order, therefore the locus of the point P is a curve of the nth order.

Note that the point $(\xi = 0, \eta = 0)$ satisfies the equation and is consequently a point of the locus. Mention is here made of this fact because in a large number of cases the point $(0, 0)$ can be shown to be an isolated point.* The origin is an $(n-1)$-fold point.

The five following equations represent the locus for one, two, three, four, and five lines, respectively:

(1) $\eta - \xi \alpha_1 - b_1 = 0,$

(2) $\eta^2 - \eta \xi (\alpha_1 + \alpha_2) + \xi^2 \alpha_1 \alpha_2 + \xi (b_1 \alpha_2 + b_2 \alpha_1) - \eta (b_1 + b_2) = 0,$

(3) $\eta^3 - \eta^2 \xi \Sigma \alpha_1 + \eta \xi^2 \Sigma \alpha_1 \alpha_2 - \xi^3 \alpha_1 \alpha_2 \alpha_3 - \eta^2 \Sigma b_1 +$
$\eta \xi [b_1 (\alpha_2 + \alpha_3) + b_2 (\alpha_1 + \alpha_3) + b_3 (\alpha_1 + \alpha_2)]$
$- \xi^2 (b_1 \alpha_2 \alpha_3 + b_2 \alpha_1 \alpha_3 + b_3 \alpha_1 \alpha_2) = 0,$

*Clebsch, *Vorlesungen ueber Geometrie*, Vol. I., pages 319-321.

$$(4)\ \eta^4-\eta^2\xi\Sigma\alpha_1+\eta^2\xi^2\Sigma\alpha_1\alpha_2-\eta\xi^3\Sigma\alpha_1\alpha_2\alpha_3+\xi^4\alpha_1\alpha_2\alpha_3\alpha_4-\eta^3\Sigma b_1+$$
$$\eta^2\xi\{b_1(\alpha_2+\alpha_3+\alpha_4)+b_2(\alpha_1+\alpha_3+\alpha_4)+b_3(\alpha_1+\alpha_2+\alpha_4)$$
$$+b_4(\alpha_1+\alpha_2+\alpha_3)\}-$$
$$\eta\xi^2\{b_1(\alpha_2\alpha_3+\alpha_2\alpha_4+\alpha_3\alpha_4)+b_2(\alpha_1\alpha_3+\alpha_1\alpha_4+\alpha_3\alpha_4)$$
$$+b_3(\alpha_1\alpha_2+\alpha_1\alpha_4+\alpha_2\alpha_4)+b_4(\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3)\}+$$
$$\xi^3(b_1\alpha_2\alpha_3\alpha_4+b_2\alpha_1\alpha_3\alpha_4+b_3\alpha_1\alpha_2\alpha_4+b_4\alpha_1\alpha_2\alpha_3)=0,$$

$$(5)\ \eta^5-\eta^4\xi\Sigma\alpha_1+\eta^3\xi^2\Sigma\alpha_1\alpha_2-\eta^2\xi^3\Sigma\alpha_1\alpha_2\alpha_3+\eta\xi^4\Sigma\alpha_1\alpha_2\alpha_3\alpha_4$$
$$-\xi^5\alpha_1\alpha_2\alpha_3\alpha_4\alpha_6-\eta^4\Sigma b_1+\eta^3\xi\Sigma b_1\alpha_2-\eta^2\xi^2\Sigma b_1\alpha_2\alpha_3$$
$$+\eta\xi^3\Sigma b_1\alpha_2\alpha_3\alpha_4-\xi^4\Sigma b_1\alpha_2\alpha_3\alpha_4\alpha_5=0.$$

By analogy it is easy to see that the general equation assumes the form

$$\eta^n-\eta^{n-1}\xi\Sigma\alpha_1+\eta^{n-2}\xi^2\Sigma\alpha_1\alpha_2-\ldots\ldots\pm\xi^n\alpha_1\alpha_2\ldots\ldots\alpha_n-$$
$$\eta^{n-1}\Sigma b_1+\eta^{n-2}\xi\Sigma b_1\alpha_2-\eta^{n-3}\xi^2\Sigma b_1\alpha_2\alpha_3+\ldots\pm\xi^{n-1}\Sigma b_1\alpha_2\alpha_3\ldots\alpha_n=0.$$

The symbolic forms

$$\begin{array}{l}\Sigma b_1\alpha_2,\\ \Sigma b_1\alpha_2\alpha_3,\\ \vdots\\ \Sigma b_1\alpha_2\alpha_3\ldots\ldots\alpha_n\end{array}$$

will be understood by comparing equations (3), (4), and (5).

2. Special applications and transformations of the above equations will be given in subsequent paragraphs.

If the n lines are parallel

$$\alpha_1=\alpha_2=\alpha_3=\ldots\ldots=\alpha_n,$$

and the resulting locus is similar to (1); and if the n lines consist of two, three, four and five sets of parallel lines the resulting loci will be similar to equations (2), (3), (4), and (5). In general the loci will differ only in having the constants

$$b_1,\ b_2,\ \ldots\ldots b_k$$

replaced respectively by the sum of the intercepts of the lines having the corresponding slopes,

$$\alpha_1,\ \alpha_2,\ \ldots\ldots\alpha_k.$$

The k systems will thus give the same locus as would the k lines which would result by considering each system separately.

If $n=2m$, *i. e.* if n is even, and the first half of the lines have the same slope as the second half then the expressions for ξ and η can be expressed by one-half the number of terms and the locus will be of the mth degree. If the lines form a regular polygon that is symmetrical with respect to the origin, n still being even, then the intercepts will be equal in pairs but of opposite sign; consequently $\xi=0$ and $\eta=0$, and the locus is the point $(0, 0)$. The same conclusions can be obtained from the general equation and also by plotting.

Equation (2) is a hyperbola and if we assume

$$\alpha_1=-\alpha_2, \text{ and } b_1=b_2=1,$$

it becomes

$$(6)\quad \eta^2-\xi^2-2\eta=0,$$

an equilateral hyperbola.

In equation (3) assume

$$\alpha_1=-\alpha_2=-\sqrt{3},\quad \alpha_3=0,$$
$$b_1=b_2=-2b3=2,$$

the three lines then form an equilateral triangle and the locus is

$$(7)\quad \eta^3-3(\xi^2\eta+\xi^2+\eta^2)=0, \text{ (see Fig. 13).}$$

In equation (5) assume

$$\begin{aligned}&\alpha_1=-\alpha_4=\tan 72^\circ,\\ &\alpha_2=-\alpha_3=\tan 144^\circ,\\ &\alpha_5=0,\\ &b_1=b_4,\\ &b_2=b_3,\\ &b_5=b_5, \text{ identical.}\end{aligned}$$

The five lines then form a regular pentagon and the origin at the center of gravity; the locus is

$$\eta^5-\eta^3\xi^2(\alpha_1^2+\alpha_2^2)+\eta\xi^4\alpha_1^2\alpha_2^2-\eta^4(2b_1+b_5+2b_5)+$$
$$\eta^2\xi^2\{2b_1\alpha_2^2+b_5-(\alpha_1^2+\alpha_2^2)+2b_2\alpha_1^2\}-\xi^4b_5\alpha_1^2\alpha_2^2=0, \text{ (see Fig. 18).}$$

3. The equation of the tangent to the curve

$$f(\xi, \eta)=0$$

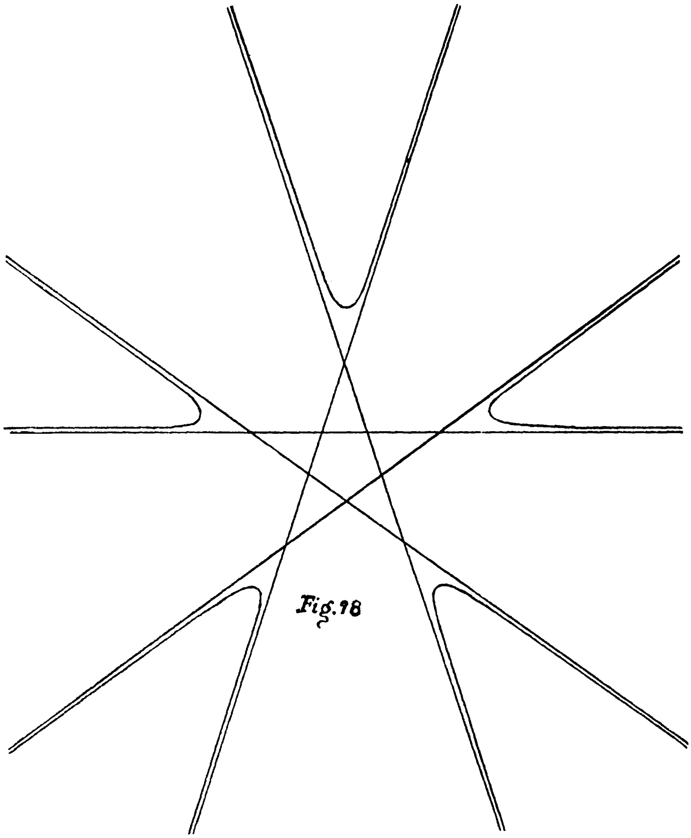

Fig. 18

$$\text{is } (x-\xi)\frac{df}{d\xi}+(y-\eta)\frac{df}{d\eta}=0,$$

$$\text{or } y=\frac{-(x-\xi)\dfrac{df}{d\xi}}{\dfrac{df}{d\eta}}+\eta.$$

For equation (7) the tangent is

$$(8) \quad y=\frac{2\xi(1+\eta)}{-\xi^2+\eta^2-2\eta}x+\frac{2\xi^2(1+\eta)}{-\xi^2+\eta^2-2\eta}+\eta.$$

The equations of the locus higher than the second degree

$$\frac{df}{d\xi} \text{ and } \frac{df}{d\eta}$$

will contain no terms independent of ξ or η and consequently the value of y in the equation of the tangent for the point (0, 0) is indeterminate. Equation (8) will serve as an application. This is a further illustration that the origin is often an isolated point.

The general method for obtaining the equation of the tangent in terms of x and y is to solve the equation

$$f(\xi, \eta)=0$$

for η in terms of ξ, there will be n values for η, and substitute these values for η in the equation of the tangent. Assign definite values to ξ and obtain the corresponding values for η in terms of ξ. Since η has n values there will be n tangents, real or imaginary.

A similar discussion applies throughout to the normal.

4. When the locus is of odd degree and symmetrical with respect to the origin linear transformations can be obtained for which the equation is invariant.

Transform the cubic

$$\eta^3-3(\xi^2\eta+\xi^2+\eta^2)=0$$

by the linear transformation

$$\xi=m_1x+c_1y,$$
$$\eta=m_2x+c_2y,$$

the result is

$$x^3(m_2^3-3m_2m_1^2)+x^2y(3m_2^2c_2-3c_2m_1^2-6m_1m_2c_1)-$$
$$3x^2(m_1^2+m_2^2)-6xy(m_1c_1+m_2c_2)+xy^2(3m_2c_2^2-3m_2c_1^2-6c_1c_2m_1)-$$
$$3y^2(c_1^2+c_2^2)+y^3(c_2^3-3c_2c_1^2)=0.$$

The following conditions are sufficient to make the two equations identical :

$$m_1^2+m_2^2=1,$$
$$c_1^2+c_2^2=1,$$
$$m_2^3-3m_2m_1^2=0,$$
$$c_2^3-3c_2c_1^2=1.$$

The solution of these equations gives the two following sets of values either of which will satisfy the condition for invariance :

$$m_1=-\tfrac{1}{2},$$
$$m_2=\tfrac{1}{2}\sqrt{3},$$
$$c_1=-\tfrac{1}{2}\sqrt{3},$$
$$c_2=-\tfrac{1}{2},$$

and

$$m_1=\tfrac{1}{2},$$
$$m_2=\tfrac{1}{2}\sqrt{3},$$
$$c_1=\tfrac{1}{2}\sqrt{3},$$
$$c_2=-\tfrac{1}{2}.$$

The corresponding substitutions are

$$(1)\quad \xi=-\tfrac{1}{2}x-\tfrac{1}{2}\sqrt{3}y,$$
$$\eta=\tfrac{1}{2}\sqrt{3}x-\tfrac{1}{2}y,$$

and

$$(2)\quad \xi=\tfrac{1}{2}x+\tfrac{1}{2}\sqrt{3}y,$$
$$\eta=\tfrac{1}{2}\sqrt{3}x-\tfrac{1}{2}y.$$

Transformation (1) corresponds to a rotation of the axes through an angle, $\theta=\frac{2\pi}{3}$, and transformation (2) is equivalent to a rotation of $\theta=\frac{\pi}{3}$ and changing y to $-y$. A transformation of this nature is known as a reflection.

From the above it follows that the cubic is invariant to the substitutions

$$\xi=x\cos\frac{2k\pi}{3}-y\sin\frac{2k\pi}{3},$$

$$\eta=x\sin\frac{2k\pi}{3}+y\cos\frac{2k\pi}{3};$$

and

$$\xi=x\cos\frac{2k'\pi}{3}+y\sin\frac{k'\pi}{3},$$

$$\eta=x\sin\frac{k'\pi}{3}+y\cos\frac{k'\pi}{3},$$

where k' is odd.

The transformations for the general case are

$$\xi=x\cos\frac{2k\pi}{n}-y\sin\frac{2k\pi}{n},$$

$$\eta=x\sin\frac{2k\pi}{n}+y\cos\frac{2k\pi}{n};$$

and

$$\xi=x\cos\frac{k'\pi}{n}+y\sin\frac{k'\pi}{n},$$

$$\eta=x\sin\frac{k'\pi}{n}-y\cos\frac{k'\pi}{n},$$

where k' is odd. The first expression for ξ and η represents rotations, and the second represents reflections.

5. Inversion.*

Let k be any point, (ξ, η) on the locus; O the origin; $\rho = \sqrt{(\xi^2+\eta^2)}$ the radius vector of the point k. With O as center and any radius, R, describe a circle. Find a point, (x, y) such that

$$\sqrt{(\xi^2+\eta^2)}.\sqrt{(x^2+y^2)}=R^2.$$

The above conditions give

$$\xi=\frac{xR^2}{x^2+y^2},\quad \eta=\frac{yR^2}{x^2+y^2}.$$

This transformation gives the inverse of the original locus which in the general case assumes the following form :

$$\prod_1^n R^2(y-x\alpha_n)=(x^2+y^2)\{b_1(y-x\alpha_2)\ldots\ldots\ldots\ldots(y-x\alpha_n)+$$
$$b_2(y-x\alpha_1)(y-x\alpha_3)\ldots.(y-x\alpha_n)+$$
$$\vdots \qquad \vdots \qquad \vdots \qquad \qquad \vdots$$
$$b_n(y-x\alpha_1)\ldots\ldots\ldots\ldots(y-x\alpha_{n-1})\}$$

This inverse is of the $(n+1)$st degree, but if the origin is changed for the general locus it will then have an absolute term and the inverse would then be of the $2n$th degree. In the derivative eqaation the origin is an $(n-1)$-fold point and an n-fold point in the inverse.

Comparison of the general equation and its inverse shows that each is invariant ta the same substitutions if (x^2+y^2) is invariant to the same. For the general linear transformation the conditions for such invariance are

$$m_1^2+m_2^2=1,$$
$$c_1^2+c_2^2=1,$$
$$m_1c_1+m_2c_2=0, \text{ or } \frac{m_1}{m_2}=-\frac{c_2}{c_1}.$$

These are the relations between the coefficients of x and y for a rotation or for a reflection.

In equation (6) change the origin so that the term -2η will disappear and assume $R=1$, the inverse will then be the Lemniscate of Bernoulli.

The inverse of equation (7) is

$$y^3-3x^2y-3(x^2+y^2)^2=0 \text{ (see Fig. 13).}$$

*Williamson's *Differential Calculus*, page 226, seventh edition.

The original locus and the latter touch in their vertices ; both are invariant to the same substitutions ; the inverse to the sides of the fundamental triangle are tangent circles to the inverse at the origin ; the inverse of the tangents at the vertices are tangent circles at the vertices of each curve and pass through the origin ; the two curves have a three-fold symmetry with respect to the lines making angles $k\pi/3$ through the origin.* (See Fig. 13 and Fig. 14.)

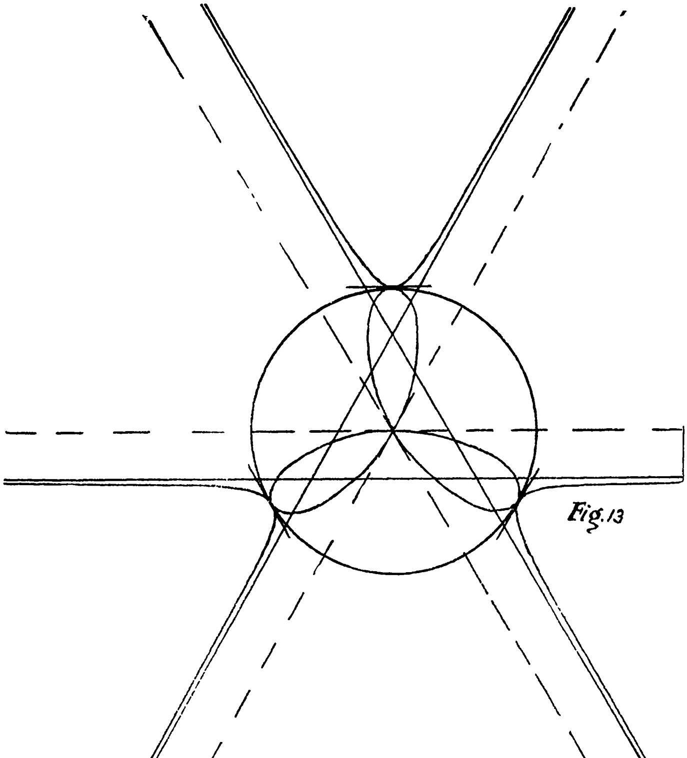

Fig. 13

6. Topological investigations.

The definition and generation of these curves is of such a nature as to render them very suitable for topological investigations. For the sake of brevity these investigations will be confined to n lines which form a polygon in the ordinary sense. The results can be extended to the most general case. The primitive origin will be defined as that origin from which the locus is determined, or through which the variable ray, $y=\lambda x$ passes.

For n lines the locus consists of n branches since the ray goes n times to infinity; $(n-1)$ of these branches pass through the origin ; curves with isolated

*These curves have also been studied by Dr. Arnold Emch of the University of Colorado, in an unpublished article of 1893.

points can be obtained only for positions of the origin within the polygon, the origin being the isolated point; when the origin is external to the polygon $(n-1)$ tangents to the locus can be drawn through it; each tangent is its own inverse; the inverse of each branch cuts the circle of reference in the same two points as does the branch itself; the tangent to each branch through the origin is

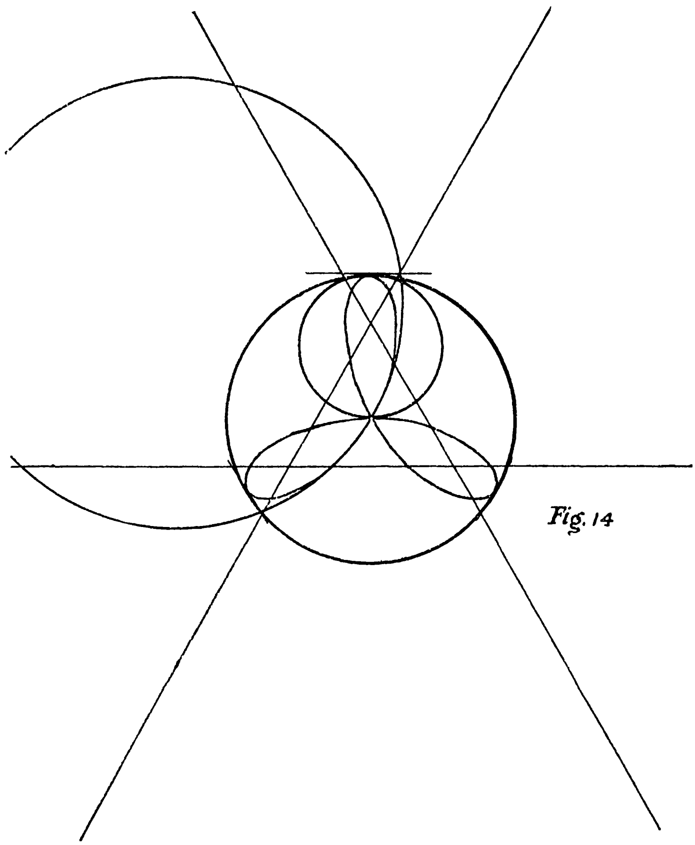

Fig. 14

an asymptote to the inverse of that branch. These statements apply only to the primitive origin and in general when it is not an isolated point. Tangents to the inverse through the origin are parallel to the asymptotes of the given locus; if the origin is not on the locus the inverse consists of n loops passing through this origin which form a closed curve. This fact is of interest because it indicates the continuity through infinity of the original locus in the sense of the theory of transformations. These statements apply to any origin.

The locus of the inverse point for the various branches differs according to the size of the circle of reference as well as for the position of the origin also when the origin is the primitive origin or otherwise.

When a number of branches intersect at the primitive origin and another

origin is then selected the inverse loops then resulting will have a common point of intersection. If the origin is so chosen that one or more of the $(n-1)$ branches through the primitive origin is concave toward it then the inverse of the independent branch will be entirely within the inverse of each and all of the other branches and vice versa if the origin is chosen within the independent branch. If the origin is so that all branches are convex toward it no inverse loop will lie within another.

For two lines one branch of the locus will always pass through the origin; for three lines the origin can be so chosen that, two branches, one branch, or no branch will pass through the origin; for four lines, three, two, or one branch will pass through the origin.

Further investigations in the thesis would seem to indicate that by properly choosing the positions of the n lines and the origin, $(n-1)$, $(n-2)$, $\ldots .1$, or no branches will pass through the origin according as n is even or odd.

Let n be odd and the intersecting lines form a regular polygon of n sides and the center of gravity be taken as primitive origin. The particular case of the equilateral triangle has been considered.

The locus consists of n branches symmetrically arranged with respect to the origin. Let the circle of reference be tangent to these branches at the vertices. The inverse is a closed curve of n symmetrical loops each of which is tangent to its derivative branch at the vertex and passes through the origin; the locus and its inverse have an n-fold symmetry with respect to the tangents to the inverse through the origin; these tangents are parallel to the sides of the polygon, are n in number and make angles of $k\pi/n$ with each other; the sides of the polygon are asymptotes to the original locus; the inverse of the sides of the polygon are tangent circles to the inverse of the origin; the inverses of the tangents at the vertices are tangent circles at the same points and pass through the origin; the locus and its inverse are invariant to the same substitutions.

If the origin is taken on a line or at the intersection of two lines the problem reduces to that of $(n-1)$ or $(n-2)$ lines. This suggests that if the origin is chosen in the immediate vicinity of such a point the resulting locus will bear a close relation to those of the $(n-1)$st or $(n-2)$nd order. Geometrical comparison of these loci and the algebraic relations between the coefficients are interesting points for investigation. A large amount of topological work has shown up many facts of a less general nature but sufficient to indicate that the problem may be considerably enlarged.